University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 13
Solutions by J. Barber

Reading:
Hand & Finch:
10.4 (Introduction; Harmonic Analysis; Hysteresis)
11 (Introduction and Overview)
11.1
11.11 (Introduction; Breaking the Symmetry; Period Doubling and the Onset of Chaos)
11.13

1.
Hand & Finch, p. 403, *Question* 7 (using symmetry). Note the typo in the statement of this problem: (b) should read $t \to t + \pi/\omega$, not $t \to t + \pi$.
Solution:
With $Q = \infty$ (no damping), equation 10.60 becomes $\ddot{q} + q + \epsilon q^3 = f \cos \omega t$.
(**a**)
Under the transformation $t \to -t$, $\ddot{q} \to \frac{1}{(-1)^2}\ddot{q} = \ddot{q}$, and $\cos \omega t \to \cos(-\omega t) = \cos \omega t$. This leaves the equation unchanged, and so Eq. 10.60 is invariant under this transformation.
(**b**)
Under the transformation $t \to t + \frac{\pi}{\omega}$, $q \to -q$, nothing happens to the time derivatives, ie $\frac{d}{dt} \to \frac{d}{dt}$. Taking this into account, the transformed equation is

$$\begin{aligned} -\ddot{q} - q + \epsilon(-q)^3 &= f \cos \omega(t + \frac{\pi}{\omega}) \\ -(\ddot{q} + q + \epsilon q^3) &= f \cos(\omega t + \pi) = -f \cos \omega t \\ \ddot{q} + q + \epsilon q^3 &= f \cos \omega t \end{aligned}$$

So 10.60 is invariant under this transformation.
Solutions to 10.60 must be invariant under the same transformations. So, if we guess a solution of the form $q(t) = \sum_{n=1}^{\infty} (A_n \cos n\omega t + B_n \sin n\omega t)$, we must have (by (**a**)):

$$\sum_{n=1}^{\infty} (A_n \cos n\omega t - B_n \sin n\omega t) = q(-t) = q(t) = \sum_{n=1}^{\infty} (A_n \cos n\omega t + B_n \sin n\omega t)$$

which implies that $\sum_{n-1}^{\infty} B_n \sin n\omega t = 0$. This can only be true for all t if all $B_n = 0$.
By (**b**):

$$\begin{aligned} -\sum_{n=1}^{\infty} A_n \cos n\omega(t + \frac{\pi}{\omega}) = -q(t + \frac{\pi}{\omega}) = q(t) &= \sum_{n=1}^{\infty} A_n \cos n\omega t \\ -\sum_{n=1}^{\infty} (-1)^n A_n \cos n\omega t &= \sum_{n=1}^{\infty} A_n \cos n\omega t \\ \sum_{n=1}^{\infty} (1 + (-1)^n) A_n \cos n\omega t &= 0 \end{aligned}$$

The quantity $1+(-1)^n$ is 2 for n even and 0 for n odd. This implies that $2\sum_{n=2,4,6...}^{\infty} A_n \cos n\omega t = 0$, which can only be true for all t if $A_n = 0$ for all even n. Thus $q(t) = \sum_{n=1,3,5...}^{\infty} A_n \cos n\omega t$.

2.
Hand & Finch Problem 10.11 (a) and (b) only (asteroid perturbed by Jupiter)
Solution:
(**a**)
The Lagrangian for this problem, taking into account the perturbation due to Jupiter, is:

$$\mathcal{L} = \frac{m_a}{2}\left(\dot{r}_a^2 + r_a^2\dot{\phi}_a^2\right) + \frac{GM_S m_a}{r_a} + \frac{GM_J m_a r_a}{r_J^2}\cos(\phi_a - \phi_J)$$

Let r_J =constant, and assume the angles ϕ_a and ϕ_J start off in phase, ie $\phi_a = \omega_a t$, $\phi_J = \omega_J t$. Then $\phi_J = \frac{\omega_J}{\omega_a}\phi_a$. If we now write the Lagrangian in terms of $u \equiv \frac{1}{r_a}$, with $\dot{r}_a = -\frac{\dot{u}}{u^2}$, we get

$$\mathcal{L} = \frac{m_a}{2}\left(\frac{\dot{u}^2}{u^4} + \frac{1}{u^2}\dot{\phi}_a^2\right) + GM_S m_a u + \frac{GM_S m_a}{r_J^2}\frac{1}{u}\cos x\phi_a$$

where $x = 1 - \frac{\omega_J}{\omega_a}$. The Euler-Lagrange equation for u is:

$$\frac{\ddot{u}}{u^4} - 2\frac{\dot{u}^2}{u^5} + \frac{1}{u^3}\dot{\phi}_a^2 + GM_S + \frac{GM_J}{r_J^2}\frac{1}{u^2}\cos x\phi_a = 0$$

Next, if we treat the effect of Jupiter as a *small* perturbation, then the asteroid's angular momentum $l = m_a\dot{\phi}_a r_a^2$ is still conserved. We can then change the independent variable from t to ϕ by using the fact that $\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \frac{lu^2}{m_a}\frac{d}{d\phi}$. Then

$$\frac{\ddot{u}}{u^4} = \frac{1}{u^4}\frac{lu^2}{m_a}\frac{d}{d\phi}\left(\frac{lu^2}{m_a}\frac{du}{d\phi}\right) = \frac{l^2}{m_a^2}\left(\frac{d^2u}{d\phi^2} + \frac{1}{u}\left(\frac{du}{d\phi}\right)^2\right)$$

and

$$\frac{\dot{u}^2}{u^5} = \frac{1}{u^5}\left(\frac{lu^2}{m_a}\frac{du}{d\phi}\right)^2 = \frac{l^2}{m_a^2 u}\left(\frac{du}{d\phi}\right)^2$$

So our equation of motion becomes

$$\frac{d^2u}{d\phi^2} + u - \frac{GM_S m_a^2}{l^2} + \frac{M_J}{M_S}\frac{GM_S m_a^2}{l^2}\left(\frac{1}{ur_J}\right)^2\cos x\phi_a = 0\ .$$

If we define $p \equiv a(1-\epsilon^2) = \frac{l^2}{GM_S m_a^2}$, then this can be written as:

$$\frac{d^2u}{d\phi^2} + u - \frac{1}{p} + \frac{M_J}{M_S}\frac{1}{p}\left(\frac{1}{ur_J}\right)^2\cos x\phi_a = 0$$

(**b**)
If we let $u = \frac{1}{p} + \delta u$, with $\delta u << \frac{1}{p}$, then our equation becomes:

$$\frac{d^2\delta u}{d\phi^2} + \delta u + \frac{1}{p}\frac{M_J}{M_S}\left(\frac{1}{\left(\frac{1}{p}+\delta u\right) r_J}\right)^2 \cos x\phi = 0$$

$$\frac{d^2\delta u}{d\phi^2} + \delta u + \frac{p}{r_J^2}\frac{M_J}{M_S}\frac{1}{(1+p\delta u)^2}\cos x\phi = 0$$

$$\frac{d^2\delta u}{d\phi^2} + \left(1 - \frac{2p^2}{r_J^2}\frac{M_J}{M_S}\cos x\phi\right)\delta u + \frac{p}{r_J}\frac{M_J}{M_S}\cos x\phi = 0$$

where in the last step we have Taylor expanded for small $p\,\delta u$. The last term is a small perturbation that averages to zero over an orbit, so we can neglect it. Furthermore, if we define $\tau \equiv \frac{x\phi}{2}$, then we obtain

$$\frac{d^2\delta u}{d\tau^2} + \left(\frac{4}{x^2} - 2\left(\frac{2p}{xr_J}\right)^2\frac{M_J}{M_S}\cos 2\tau\right)\delta u = 0$$

$$\frac{d^2\delta u}{d\tau^2} + (a - 2q\cos 2\tau)\,\delta u = 0$$

where a and q are as defined in the problem.

3.
Using Hand & Finch Eqs. (9.18) and (9.19), derive Eq. (9.20).
Solution:

$$x_1 = l\sin\phi_1 \quad\rightarrow\quad \dot{x}_1 = l\dot{\phi}_1\cos\phi_1 \qquad y_1 = -l\cos\phi_1 \quad\rightarrow\quad \dot{y}_1 = l\dot{\phi}_1\sin\phi_1$$
$$x_2 = x_1 + l\sin(\phi_1+\phi_2) \quad\rightarrow\quad \dot{x}_2 = l\dot{\phi}_1\cos\phi_1 + l(\dot{\phi}_1+\dot{\phi}_2)\cos(\phi_1+\phi_2)$$
$$y_2 = y_1 - l\cos(\phi_1+\phi_2) \quad\rightarrow\quad \dot{y}_2 = l\dot{\phi}_1\sin\phi_1 + l(\dot{\phi}_1+\dot{\phi}_2)\sin(\phi_1+\phi_2)$$

$$\begin{aligned} V = mg(y_1+y_2) &= mg\left(-l\cos\phi_1 - l\cos\phi_1 - l\cos(\phi_1+\phi_2)\right) \\ &= -mgl\left(2\cos\phi_1 + \cos(\phi_1+\phi_2)\right) \end{aligned}$$

$$\begin{aligned} T = \frac{m}{2}\Big(&(l\dot{\phi}_1\cos\phi_1)^2 + (l\dot{\phi}_1\sin\phi_1)^2 + (l\dot{\phi}_1\cos\phi_1 + l(\dot{\phi}_1+\dot{\phi}_2)\cos(\phi_1+\phi_2))^2 \\ &+ (l\dot{\phi}_1\sin\phi_1 + l(\dot{\phi}_1+\dot{\phi}_2)\sin(\phi_1+\phi_2))^2\Big) \\ = \frac{ml^2}{2}\Big(&\dot{\phi}_1^2(\cos^2\phi_1+\sin^2\phi_1) + \dot{\phi}_1^2(\cos^2\phi_1+\sin^2\phi_1) \\ &+ 2\dot{\phi}_1(\dot{\phi}_1+\dot{\phi}_2)(\cos\phi_1\cos(\phi_1+\phi_2) + \sin\phi_1\sin(\phi_1+\phi_2)) \\ &+ (\dot{\phi}_1+\dot{\phi}_2)^2(\cos^2(\phi_1+\phi_2)+\sin^2(\phi_1+\phi_2))\Big) \\ = \frac{ml^2}{2}&\left(2\dot{\phi}_1^2 + 2\dot{\phi}_1(\dot{\phi}_1+\dot{\phi}_2)\cos\phi_2 + (\dot{\phi}_1+\dot{\phi}_2)^2\right) \end{aligned}$$

4.
Hand & Finch, p. 427, *Question* 1 (Hamiltonian)
Solution:
We make the change of notation $\phi_1 \to \alpha$ and $\phi_2 \to \beta$. Let $\mathcal{L}' = \frac{\mathcal{L}}{mgl}$ and let $t \to \sqrt{\frac{g}{l}}\, t$. Then:

$$\mathcal{L}' = \frac{T-V}{mgl} = \frac{1}{2}\left(2\dot{\alpha}^2 + 2\dot{\alpha}(\dot{\alpha}+\dot{\beta})\cos\beta + (\dot{\alpha}+\dot{\beta})^2\right) + 2\cos\alpha + \cos(\alpha+\beta)$$

Find conjugate momenta:

$$\begin{aligned} l_\alpha &= \frac{\partial \mathcal{L}'}{\partial \dot{\alpha}} = 2\dot{\alpha} + (2\dot{\alpha}+\dot{\beta})\cos\beta + \dot{\alpha} + \dot{\beta} \\ &= (3+2\cos\beta)\dot{\alpha} + (1+\cos\beta)\dot{\beta} \\ l_\beta &= \frac{\partial \mathcal{L}'}{\partial \dot{\beta}} = \dot{\alpha}\cos\beta + \dot{\alpha} + \dot{\beta} \\ &= (1+\cos\beta)\dot{\alpha} + \dot{\beta} \end{aligned}$$

From these one can solve for $\dot{\alpha}$ and $\dot{\beta}$ to yield:

$$\begin{aligned} \dot{\alpha} &= \frac{l_\alpha - (1+\cos\beta)l_\beta}{3+2\cos\beta-(1+\cos\beta)^2} = 2\frac{l_\alpha-(1+\cos\beta)l_\beta}{3-2\cos 2\beta} \\ \dot{\beta} &= \frac{(3+2\cos\beta)l_\beta - (1+\cos\beta)l_\alpha}{3+2\cos\beta-(1+\cos\beta)^2} = 2\frac{(3+2\cos\beta)l_\beta-(1+\cos\beta)l_\alpha}{3-2\cos 2\beta} \end{aligned}$$

The Lagrangian can now be rewritten as

$$\begin{aligned} \mathcal{L}' &= \frac{1}{2}\left(\left((3+2\cos\beta)\dot{\alpha} + (1+\cos\beta)\dot{\beta}\right)\dot{\alpha} + \left((1+\cos\beta)\dot{\alpha}+\dot{\beta}\right)\dot{\beta}\right) + 2\cos\alpha + \cos(\alpha+\beta) \\ &= \frac{1}{2}l_\alpha\dot{\alpha} + \frac{1}{2}l_\beta\dot{\beta} + 2\cos\alpha + \cos(\alpha+\beta) \end{aligned}$$

Using the definition of the Hamiltonian and our expressions for $\dot{\alpha}$ and $\dot{\beta}$, we can now get

$$\begin{aligned} \mathcal{H} \equiv l_\alpha\dot{\alpha} + l_\beta\dot{\beta} - \mathcal{L}' &= \frac{1}{2}l_\alpha\dot{\alpha} + \frac{1}{2}l_\beta\dot{\beta} - 2\cos\alpha - \cos(\alpha+\beta) \\ &= \frac{1}{2}l_\alpha\left(2\frac{l_\alpha-(1+\cos\beta)l_\beta}{3-2\cos\beta}\right) + \frac{1}{2}l_\beta\left(2\frac{(3+2\cos\beta)l_\beta-(1+\cos\beta)l_\alpha}{3-2\cos\beta}\right) - 2\cos\alpha - \cos(\alpha+\beta) \\ &= \frac{l_\alpha^2 - 2(1+\cos\beta)l_\alpha l_\beta + (3+2\cos\beta)l_\beta^2}{3-2\cos\beta} - 2\cos\alpha - \cos(\alpha+\beta) \end{aligned}$$

5.
Hand & Finch, p. 454, *Question* 12 (phase space flow equations)
Solution:
If the force (the *torque*, actually!) is $F\sin\omega t$, then from elementary considerations the equation of motion is

$$\begin{aligned} ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta &= F\sin\omega t \\ \ddot{\theta} + \frac{b}{ml^2}\dot{\theta} + \frac{g}{l}\sin\theta &= \frac{F}{ml^2}\sin\omega t \end{aligned}$$

Now define new constants: $\frac{1}{Q} \equiv \frac{b}{ml^2}$, $\omega_o^2 \equiv \frac{g}{l}$, and $f \equiv \frac{F}{ml^2}$. We then have:

$$\ddot{\theta} + \frac{\dot{\theta}}{Q} + \omega_o^2 sin\theta = f \sin \omega t$$

If we define $p \equiv \dot{\theta}$, then we can write

$$\begin{cases} \dot{\theta} = p \\ \dot{p} + \frac{p}{Q} + \omega_o^2 \sin\theta = f \sin \omega t \end{cases}$$

which is identical to Eq. 11.22.

6.
Hand & Finch, p. 455, *Question* 13 (symmetry breaking)
Solution:
If $t \to t + \frac{\pi}{\omega}$, $\frac{d}{dt} \to \frac{d}{dt}$ (derivatives are unaffected). So, with $\theta \to -\theta$ and $p \to -p$, eqn 11.22 becomes:

$$\begin{cases} -\dot{\theta} = -p \\ -\dot{p} = -\omega_o^2 \sin(-\theta) - \frac{-p}{Q} + f \sin(\omega(t + \frac{\pi}{\omega})) \end{cases}$$

$$\begin{cases} \dot{\theta} = p \\ \dot{p} = -\omega_o^2 \sin\theta - \frac{p}{Q} - f \sin(\omega t + \pi) \end{cases}$$

$$\begin{cases} \dot{\theta} = p \\ \dot{p} = -\omega_o^2 \sin\theta - \frac{p}{Q} + f \sin \omega t \end{cases}$$

which is identical to the original equations. Eqn 11.22 is thus invariant under this transformation.