Homework #5: Hints

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1 Particle in an EM field

Write down the equation of motion, using complex notation (i.e., replace $\cos\omega t$ with $e^{i\omega t}$). Solve by guessing the form of the solution and plugging into the equation. Note that you are asked to find the *steady state* solution – this means the particular solution, not the homogeneous solution. That's why the amplitude and phase are not adjustable, and why you can determine them from the parameters given.

2 The Damped, Undriven oscillator

(a)

Write down (derive or copy from the notes) the general solution to the undriven, damped oscillator. Calculate $E(t) = T(t) + U(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2$. Find t' such that $E(t') = \frac{1}{e}E(0)$, and calculate N, the number of periods, from t'.

(b)

First, calculate the energy stored in the oscillator by solving the particular equation (use the method we used in (1) of assuming the form of the solution with adjustable parameters for the amplitude and phase, and plugging into the equation to find them). You can assume a driving force of the form $F = F_0 \cos\omega t = \Re(F_0 e^{i\omega t})$. Next, calculate the work done by the driving force in one period by calculating:

$$W = \oint F dx = \int_{t_0}^{t_0+2\pi\omega} F(t)v(t)dt$$

You should know $v(t)$ by differentiating $x(t)$ that you found above.

3 Specific Solutions to a Damped Oscillator

Both parts are solved almost identically. Start by writing down the general solution to the undriven oscillator, either underdamped or overdamped (this is the part that's different between the two solutions. Find the solutions in the notes if you can't rederive them; you should feel free to use anything given in the lecture notes unless specifically told to rederive it). Now, just plug in the values that you're given for Q to find either ω_γ or γ in terms of ω_0. You should have two adjustable constants left; use the boundary conditions given in the problem to fix those, and write out the final solution with no parameters except x_0, ω_0, and t.

4 The Critically Damped Oscillator

Start, as usual, by writing out the general (particular) equation for a damped, driven oscillator, in complex form. Assume the same form of the solution that we always do, and solve for the amplitude and phase of the particular solution $x_p(t)$. Now, write out the homogeneous equation, and either solve it for $Q = \frac{1}{2}$ or use the overdamped solution with the substitution $\gamma = 2\omega_0$. Add this solution ($x_h(t)$) and the particular solution, and fix the constants using boundary conditions. $x(t)$ should have no constants in it except G and ω_0. NOTE: you *are* given $\dot{x}(0)$ in the problem, if you think about it.

5 Woofer Design

(a)

What happens if you displace an overdamped oscillator? What about if it's an underdamped oscillator? Does this lead you to an answer?

(b)

Write down the given equation of motion, and divide through by m to get it in familiar form. Identify γ and ω_0. You are given ν_0; use that to find ω_0. That, together with Q, should give you a value for γ.

As usual, find the amplitude as a function of various parameters using the particular equation. The key insight is that the sound intensity is proportional to the square of the sound *amplitude* – and the sound amplitude is proportional to the force exerted by the cone on the air around it. What is the force exerted by the cone on the air proportional to? That should give

you the sound intensity (to within a constant factor) in terms of ω and A. Now, just calculate the ratio $I(\omega)/I(\omega_0)$ for the given frequencies.

(c)

Basically, what you should do is try to determine how ω_0 and γ depend on the cone area A. To find this, consider how m,k, and b depend on A. You should find that one of $\{\omega_0, \gamma\}$ is independent and one is proportional to A^{-1}. Then define $z = \frac{A}{A_{\text{initial}}}$, plug the z-dependence of ω_0 and γ into the expression for intensity from the previous section, and plot it for various values of z. At what values of z does the curve that you get look "smooth" – that is, has no spikes or small spikes?

6 Fourier Series

First of all, you must assume that the function is periodic, although the definition given in the problem doesn't make it look that way. That is, $F\left(t + \frac{4\pi}{\omega}\right) = F(t)$.

Next, find the period of the function. It's basically given to you in the hint above. Now, write out equation (3.12) from the reader, and use equation (3.13) to find the coefficients f_n and g_n. You're on your own doing the integrals; I recommend looking them up or using a computer program like Maple or Mathematica. Once you have the coefficients, just write out equation (3.12) with the coefficients in place, and you're done.

7 The Funkily Driven Oscillator

Keep in mind that you need only the particular solution, not the homogenous solution (transients have been damped out).

Start by writing out the particular equation, with the whole nasty force term on the right (in the Fourier form given in problem [6]). You should have a driving term:

$$\frac{F_0}{2m}\sin\omega t + \sum_{n \text{ odd}} \frac{4}{\pi(4-n^2)}\frac{F_0}{m}\cos\frac{n}{2}\omega t$$

Use that form if you didn't get the right answer for (6), or go back and fix it in (6). As I explained in discussion section (for those of you who were there), the solution to a sum of driving terms is simply the sum of the solutions

to the individual driving terms, so you can write the solution as $x(t) =$ the real part of a sum of complex exponentials, where the frequency of each complex exponential corresponds to the frequency of a driving term given in the expression above. You should have one term corresponding to the first ($\sin \omega t$) term above, and a sum of terms indexed by n corresponding to the sum of driving terms above. Each term in the solution should have its own independent amplitude, perhaps A and B_n.

Now, solve the equation separately for each ω in the sum. Remember to do this all in the complex exponential formulation, and write your final answer as the real part of the complex solution that you get. The final answer is heinously long and nasty; please write it neatly so that I can check to make sure it's right or correct it!

8 Overdamped Green's Function

The Green's Function is simply the solution to the particular equation with a δ function driving term. That is, it's the solution to the equation:

$$\ddot{X}_g(t) + \gamma \dot{X}_g(t) + \omega_0^2 X_g(t) = \delta(t)$$

For those of you who don't feel comfortable with delta functions, $\delta(t)$ spikes at $t = 0$ and is zero everywhere else; it's infinite at the origin, but in such a way that $\int_{-\infty}^{\infty} \delta(t) dt \equiv 1$.

For all $t > 0$, the driving term is zero, so the particular equation becomes the homogeneous equation there. Why don't you have to worry about $t < 0$?

When you solve the homogeneous equation for $t > 0$, you find there are undetermined constants. You should solve for those using the initial conditions $X_g(0)$ and $\dot{X}_g(0)$ – keeping in mind that you really want to use the values of X_g and its derivative just *after* $t = 0$ – that is, after the delta function drive occurs. Use the concept of *impulse* to find the velocity immediately after the "kick" is applied at $t = 0$. Once you've matched these conditions, you're done.