University of California, Berkeley
Physics 110A Fall 2003 Section 2 (*Strovink*)

Solutions to Practice Problems (these are Daniel Larson's solutions)

1. **Griffiths 1.38** The divergence theorem tells us that the integral of the divergence over the volume equals the integral of the vector field over the surface. So we need to compute two integrals in each case.

   (a) $\boldsymbol{\nabla}\cdot(r^2\,\hat{\mathbf{r}}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^4) = 4r$, where I used equation (1.71). Now, $\int(\boldsymbol{\nabla}\cdot\mathbf{v}_1)\,d\tau = \int(4r)r^2\,dr\,\sin\theta\,d\theta\,d\phi = \left(\int_0^R 4r^3\,dr\right)\left(\int d\Omega\right) = r^4\Big|_0^R(4\pi) = 4\pi R^4$. On the other hand, on the surface of the sphere, $r = R$, and so $\int\mathbf{v}_1\cdot d\mathbf{a} = \int(R^2\,\hat{\mathbf{r}})\cdot(R^2\sin\theta\,d\theta, d\phi\,\hat{\mathbf{r}}) = R^4\int d\Omega = 4\pi R^4$. The two integrals agree.

   (b) $\boldsymbol{\nabla}\cdot(1/r^2)\,\hat{\mathbf{r}} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{1}{r^2}) = 0$, so $\int(\boldsymbol{\nabla}\cdot\mathbf{v}_2)\,d\tau = 0$. On the other hand, on the surface of a sphere radius R, $\int\mathbf{v}_2\cdot d\mathbf{a} = \int(\frac{1}{R^2}\,\hat{\mathbf{r}})\cdot(R^2 d\Omega\,\hat{\mathbf{r}}) = \int d\Omega = 4\pi$. The two integrals *don't* agree! The reason is that $\boldsymbol{\nabla}\cdot\mathbf{v}_2 = 0$ except at the origin, where it becomes infinite. Thus our calculation of $\int(\boldsymbol{\nabla}\cdot\mathbf{v}_2)\,d\tau$ is incorrect. We'll learn how to fix this using the Dirac delta-function. The correct answer is 4π, whic is what we got the using the surface integral because that method avoids the problem at the origin.

2. **Griffiths 1.46**

   (a) Since the charge is only at the specific point $\mathbf{r}'$ we will need to use a delta-function. $\rho(\mathbf{r}) = q\delta^3(\mathbf{r}-\mathbf{r}')$. The volume integral is then $\int\rho(\mathbf{r})\,d\tau = q\int\delta^3(\mathbf{r}-\mathbf{r}')\,d\tau = q$ as it should be.

   (b) The electric dipole is just two different point charges at different places. $\rho(\mathbf{r}) = q\delta^3(\mathbf{r}-\mathbf{a}) - q\delta^3(\mathbf{r})$.

   (c) There is no charge anywhere except where $r = R$, so $\rho(\mathbf{r}) = A\delta(r-R)$. We need to integrate over all space to find A. $Q = \int\rho\,d\tau = \int A\delta(r-R)4\pi r^2\,dr = A4\pi R^2$. So $A = \frac{Q}{4\pi R^2}$. Thus $\rho(r) = \frac{Q}{4\pi R^2}\delta(r-R)$.

3. **Griffiths 2.6** The first thing to notice is that any horizontal components of the electric field will cancel because

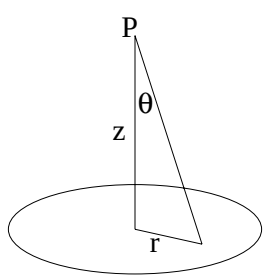


Figure 1: Problem 7.

of the symmetry of the disk. So the resulting electric field will be in the $\hat{\mathbf{z}}$ direction. Then we just need to add up the contributions of the z-components due to every point on the disk. For a generic point on the disk located at a distance r from the center, the distance to the point P is $\sqrt{r^2+z^2}$. Thus the z-component of the E field at P due to that point is $\frac{1}{4\pi\epsilon_0}\frac{dq}{r^2+z^2}\cos\theta$ where θ is the angle between the line connecting P with the center of the disk and the line connecting P with the generic element of charge on the disk (see Fig. 1). So we see $\cos\theta = \frac{z}{\sqrt{r^2+z^2}}$. To get the total E-field at P we need to add up these contributions for every little element of charge on the disk. The amount of charge in a small area element is $dq = \sigma r dr d\phi$. Thus

$$\begin{aligned} E &= \int_0^R\int_0^{2\pi}\frac{1}{4\pi\epsilon_0}\frac{z\sigma r dr\,d\phi}{(r^2+z^2)^{3/2}}\,\hat{\mathbf{z}} = \frac{2\pi z\sigma}{4\pi\epsilon_0}\int_0^R\frac{r dr}{(r^2+z^2)^{3/2}}\,\hat{\mathbf{z}} \\ &= \frac{2\pi z\sigma}{4\pi\epsilon_0}\left[-(r^2+z^2)^{-1/2}\right]_0^R\hat{\mathbf{z}} = \frac{z\sigma}{2\epsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right]\hat{\mathbf{z}}. \end{aligned}$$

Finally, it is always good to check the limiting cases of your results. For $R \to \infty$ the second term vanishes and we're left with $\sigma/2\epsilon_0$ which is the electic field due to an infinite sheet of charge. For $z \gg R$ at first glance the two terms in brackets cancel, so we'd get zero, which is true, but not so informative. To get a more precise estimate of how the field falls off for large z we need to expand the square root.

$$\frac{1}{\sqrt{R^2+z^2}} = \left[z^2\left(1+\frac{R^2}{z^2}\right)\right]^{-1/2} = \frac{1}{z}\left(1-\frac{R^2}{2z^2}+\mathcal{O}\left(\frac{R^4}{z^4}\right)\right)$$

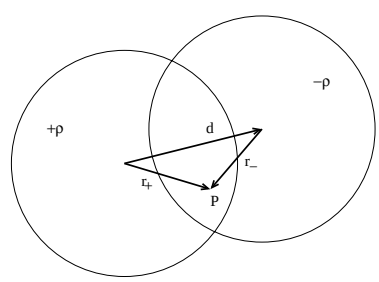
Since $z \gg R$, we can ignore the R^4/z^4 piece. Plugging the expansion back into the formula for the electric field, we find $E \to \frac{\sigma\pi R^2}{4\pi\epsilon_0 z^2}$ which is the field a distance z away from a point charge with $Q = \sigma\pi R^2$.

4. **Griffiths 2.18** First we need the electric field inside a uniformly charged sphere. Imagining a spherical Gaussian surface of radius r inside the charged sphere, symmetry tells us that the electric field must be pointing radially outward and have the same magnitude over the whole surface. So $\int \mathbf{E}(r)\cdot d\mathbf{a} = 4\pi r^2 E(r) = Q_{\text{enc}}/\epsilon_0 = \rho 4\pi r^3/3$. Thus $\mathbf{E}(r) = \rho r/(3\epsilon_0)\,\hat{\mathbf{r}} = \rho r/(3\epsilon_0)\,\mathbf{r}$ where $\mathbf{r}$ is the unit vector from the center of the charged sphere to the point in question.

   In this problem we have two charged spheres with the vector $\mathbf{d}$ pointing from the center of the positive sphere to the center of the negative sphere. For a point P in the region of overlap, there will be a contribution to the E-field from both spheres. If we let $\mathbf{r}_+$ be the vector from the center of the positive sphere to P and $\mathbf{r}_-$ be the vector from the center of the negative sphere to P, then the total E-field at P is

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho}{3\epsilon_0}\mathbf{r}_+ + \frac{-\rho}{3\epsilon_0}\mathbf{r}_- = \frac{\rho}{3\epsilon_0}(\mathbf{r}_+ - \mathbf{r}_-).$$

   But from the figure, we see that $\mathbf{r}_+ - \mathbf{r}_- = \mathbf{d}$. So the total electric field in the overlap region is $\mathbf{E} = \rho/3\epsilon_0\,\mathbf{d}$.



Problem 1

5. **Griffiths 2.36 (a-c)**

   (a) For each of the cavities, we may imagine a Gaussian surface that is completely in the conductor and surrounds the cavity. Since there is no electric field in the metal of the conductor, $\int \mathbf{E}\cdot d\mathbf{a} = 0$. By Gauss's Law this means that the charge enclosed must be zero, so the total charge on the inner surface of the cavity must be exactly opposite of the point charge contained in the cavity. By symmetry, there is no reason for the surface charge to be anything but uniformly distributed. Thus $\sigma_a = q_a/4\pi a^2$ and $\sigma_b = q_b/4\pi b^2$. Since the conductor is neutral, the charge on the outer surface must be opposite the charge on the inner surface, and again it will be uniformly distributed, so $\sigma_R = (q_a+q_b)/4\pi R^2$.

   (b) To find the field outside the conductor, the argument is exactly the same as in Example 2.9 in the text. The conductor makes the electric field outside look exactly like two point charges q_a and q_b at the origin. So $\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_a+q_b}{r^2}\,\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector from the center of the conducting sphere.

   (c) The surface charge in cavity a cancels the electric field due to the point charge q_a everywhere outside the cavity. So the only source of electric field in cavity a is the point charge q_a and the surface charge.

But using Gauss's Law and the spherical symmetry, the electric field inside the cavity is just that of the point charge q_a, namely $\mathbf{E}_a = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_a}{r_a^2}\,\hat{\mathbf{r}}_a$, where $\mathbf{r}_a$ is a unit vector from the center of cavity a. The same reasoning applies to cavity b, so $\mathbf{E}_b = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_b}{r_b^2}\,\hat{\mathbf{r}}_b$.

6. **Griffiths 3.6** The xy plane is a grounded conductor, so it is at zero potential. We can reproduce this situation by considering a similar setup without the conductor, but instead with a charge $+2q$ at $z = -d$ and a charge $-q$ at $z = -3d$. These image charges make the potential $V = 0$ anywhere in the xy plane, so it exactly matches the boundary conditions in the original problem with the conductor. The force on the charge $+q$ is then given by Coulomb's Law:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\left(\frac{(q)(-q)}{(6d)^2} + \frac{(q)(2q)}{(4d)^2} + \frac{(q)(-2q)}{(2d)^2}\right)\hat{\mathbf{z}} = \frac{1}{4\pi\epsilon_0}\left(\frac{29q^2}{72d^2}\right)\hat{\mathbf{z}}.$$

7. **Griffiths 4.15**

(a)

$$\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2}; \quad \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} \mathbf{P}\cdot\hat{\mathbf{r}} = k/b & (\text{at } r = b), \\ \mathbf{P}\cdot(-\hat{\mathbf{r}}) = -k/a & (\text{at } r = a). \end{cases}$$

The spherical symmetry again tells us that $\mathbf{E}$ is radial. Thus $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{r^2}\,\hat{\mathbf{r}}$ in all three regions. For a gaussian surface with $r < a$ there is no charge enclosed, so $\mathbf{E}(r < a) = 0$. For $a < r < b$ the charge enclosed is $\left(\frac{-k}{a}\right)4\pi a^2 + \int_a^r \left(\frac{-k}{r'^2}\right)4\pi r'^2\,dr' = -4\pi ka - 4\pi k(r-a) = -4\pi kr$, so $\mathbf{E}(a < r < b) = -(k/\epsilon_0 r)\,\hat{\mathbf{r}}$. Finally, for $r > b$ the charge inclosed is the same as in the previous calculation (with $r = b$) plus the surface charge at $r = b$. So $Q_{\text{enc}} = -4\pi kb + 4\pi b^2(k/b) = 0$, thus $\mathbf{E}(r > b) = 0$.

(b) The spherical symmetry tells us $\mathbf{D}$ must be radial, so $\int \mathbf{D}\cdot d\mathbf{a} = 4\pi r^2 D(r)$ at some radius r. But since there is no free charge anywhere, we must have $bfd = 0$ everywhere. Since $\epsilon_0\mathbf{E} = \mathbf{D} - \mathbf{P} = -\mathbf{P}$, since $\mathbf{P} = 0$ both inside and outside the shell, $\mathbf{E} = 0$ both outside and inside the shell. Within the shell, $\mathbf{E} = -P/\epsilon_0 = -(k/\epsilon_0 r)\,\hat{\mathbf{r}}$. This agrees with part (a) and was far quicker.

8. **Griffiths 4.18** Choose coordinates so that the capacitor is in the xy-plane and $\hat{\mathbf{z}}$ points "up" from the negatively charged plate towards the positively charged plate. Let's start this problem by thinking physically about what will happen. There is free charge placed on the top and bottom plates, which will produce some electric field pointing down. (We will assume the capacitor is big enough in the xy-directions so that the electric field will be only in the z-direction.) But that electric field will polarize the two dielectrics, producing $\mathbf{P}$ pointing in the same direction as $\mathbf{E}$, which in turn induces positive bound surface charge on the bottoms of each dielectric surface and negative bound charge on the top of each dielectric surface. These collections of bound charge will also produce their own electric field, which we also need to take into account. Now let's work through the details.

(a) The $\mathbf{D}$ field depends only on the free charge, so with $+\sigma$ on the top plate and $-\sigma$ on the bottom plate, the $\mathbf{D}$ field in between the plates will be $\mathbf{D} = \sigma(-\hat{\mathbf{z}})$, which is the $\mathbf{D}$ field between two infinite planes with free surface charges $\pm\sigma$. It has the same value in each of the slabs.

(b) Since we're dealing with linear dielectrics, $\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E}$. However, ϵ_r is different in the two slabs. Thus in slab 1, $\mathbf{E}_1 = \mathbf{D}/\epsilon_0\epsilon_r^{(1)} = -\sigma/2\epsilon_0\,\hat{\mathbf{z}}$, and in slab 2, $\mathbf{E}_2 = \mathbf{D}/\epsilon_0\epsilon_r^{(2)} = -\sigma/1.5\epsilon_0\,\hat{\mathbf{z}} = -2\sigma/3\epsilon_0\,\hat{\mathbf{z}}$.

(c) In a linear dielectric, $\mathbf{P} = \epsilon_0(\epsilon_r - 1)\mathbf{E}$, so in slab 1 we have $\mathbf{P}_1 = \epsilon_0(2-1)\mathbf{E}_1 = -\sigma/2\,\hat{\mathbf{z}}$ and in slab 2 we have $\mathbf{P}_2 = \epsilon_0(1.5-1)\mathbf{E}_2 = -\sigma/3\,\hat{\mathbf{z}}$.

(d) We find the potential difference by integrating $\mathbf{E}\cdot d\ell$ between the two plates. Since $\mathbf{E}$ is uniform in each of the slabs, and points straight down, we get $V = E_1 a + E_2 a = 7a\sigma/6\epsilon_0$.

(e) $\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P} = 0$ in both slabs. $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$, so remembering that $\hat{\mathbf{n}}$ always points out of each slab, $\sigma_b^{(1)} = +\sigma/2$ at the bottom of slab 1 and minus that at the top of slab 1. $\sigma_b^{(2)} = +\sigma/3$ at the bottom of slab 2 and minus that at the top. See the figure below.

(f) Using all of the charges, we want to recalculate the electric field in each slab. All of the charges are surface charges distributed on (approximately) infinite planes of charge, so we use the result that the electric field due to a single plane of surface charge doesn't depend on the distance from the plane. Inside slab 1 it is as if there was a single plane on top with net surface charge equal to the free charge on the top capacitor plate plus the bound charge on the top of slab 1, namely $\sigma - \sigma/2 = \sigma/2$; and also a single plane below with net surface charge dues to the bound charge on the bottom of slab 1, the top of slab 2, and the bottom of slab 2, and the free charge on the bottom capacitor plate, namely $\sigma/2 - \sigma/3 + \sigma/3 - \sigma = -\sigma/2$. So inside slab 1 the electric field is the same as between two infinite plates with surface charge $\pm\sigma/2$, so $E_1 = \sigma/2\epsilon_0$ (pointing down). Similarly, inside slab 2 there is net surface charge $\sigma - \sigma/2 + \sigma/2 - \sigma/3 = 2\sigma/3$ above and $\sigma/3 - \sigma = -2\sigma/3$ below. So the electric field in slab 2 is $E_2 = 2\sigma/3\epsilon_0$, again pointing down.

Problem 8. Griffiths 3.18