1 **Set 11**

a) From substitution of $\vec{G} = V\vec{\nabla}U$ into divergence theorem,

$$\int \vec{\nabla}\cdot(V\vec{\nabla}U)\,d\tau = \oint (V\vec{\nabla}U)\cdot\hat{n}\,da$$

$$\boxed{\text{using } \nabla\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)}$$

$$\vec{\nabla}U = \frac{\partial U}{\partial n}\hat{n}$$

$$V(\nabla^2 U) + \vec{\nabla}U\cdot\vec{\nabla}V$$

$$\therefore \int (\nabla V\cdot\nabla U + V\nabla^2 U)\,d\tau = \oint V\frac{\partial U}{\partial n}\,da$$

b) Switching $V \leftrightarrow U$ in the above, and subtracting from each other, we obtain

$$\int (V\nabla^2 U - U\nabla^2 V)\,d\tau = \oint \left(V\frac{\partial U}{\partial n} - U\frac{\partial V}{\partial n}\right)da$$

c) substituting $\nabla^2 U = -k^2 U$ & $\nabla^2 V = -k^2 V$, L.H.S. vanishes.

$\boxed{2}$ (a)

Green's theorem now gives

$$0 = \oint_{\text{outer surface}} V_0 \cancel{e^{i\omega t}} \left(\frac{e^{ikr}}{r} \frac{\partial U}{\partial n} - U \frac{\partial}{\partial n} \frac{e^{ikr}}{r} \right) da$$

$$+ \oint_{r=R} V_0 \cancel{e^{i\omega t}} \left(\frac{e^{ikr}}{r} \frac{\partial U}{\partial(-r)} - U \frac{\partial}{\partial(-r)} \frac{e^{ikr}}{r} \right) R^2 d\Omega$$

The derivative term in inner surface integral part is

$$\frac{\partial}{\partial(-r)} \frac{e^{ikr}}{r} = -\left(ik - \frac{1}{r}\right) \frac{e^{ikr}}{r}$$

For $R \to 0$, $\frac{1}{r}$ term dominates and

$$\oint_{r=R} -U \frac{1}{R^2} e^{ik(0)} R^2 d\Omega = -4\pi U(0)$$

Hence we obtain

$$U(0) = \frac{1}{4\pi} \oint \left(\frac{e^{ikr}}{r} \frac{\partial U}{\partial n} - U \frac{\partial}{\partial n} \frac{e^{ikr}}{r} \right) da$$

$\boxed{2}$ (b)

Substituting $U = U_0 \dfrac{e^{i(kr'-\omega t)}}{r'}$ into

K. I. T.,

$$U(P) = \frac{U_0 e^{-i\omega t}}{4\pi} \iint_{\text{aperture}} \left(\frac{e^{ikr}}{r} \frac{\partial}{\partial n} \frac{e^{ikr'}}{r'} - \frac{e^{ikr'}}{r'} \frac{\partial}{\partial n} \frac{e^{ikr}}{r} \right) d$$

Now the derivative term becomes

$$\frac{\partial}{\partial n} \frac{e^{ikr'}}{r'} = \vec{\nabla} \frac{e^{ikr'}}{r'} \cdot \hat{n} \quad \text{(directional derivative)}$$

$$= \left(ik - \frac{1}{r'} \right) \frac{e^{ikr'}}{r'} \hat{r}' \cdot \hat{n}$$

For far zone limit, ik term dominates and

$$U(P) = \frac{-ik U_0 e^{-i\omega t}}{4\pi} \int_{ap} \frac{e^{ik(r+r')}}{r r'} (\hat{r} \cdot \hat{n} - \hat{r}' \cdot \hat{n}) \, da$$

13] From the symmetry in the integral over aperture

$$\int_{ap} \otimes = \int_{-\infty}^{\infty} dx \int_{\infty}^{\infty} dy \; \otimes$$

$$= 2 \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dy \; \otimes$$

in Fresnel-Kirchoff integral formula, we can easily see

$$U_{w/\,screen} = \frac{1}{2} U_{w/out\ screen} .$$

So $$\frac{I}{I_0} = \left(\frac{1}{2}\right)^2 = \frac{1}{4} .$$

(also from Babinet's principle (which is a consequence of FKIF), easy to see $U_{w/screen} = \frac{1}{2} U_{w/out\ screen}$ from $U_{ap} + U_{comp} = U_{no}$.)

$\boxed{4}$

(a)

$$f \otimes g = \int_{-\infty}^{\infty} dx' \, f(x') \, g(x-x')$$

$x - x' = y'$

$$= \int_{+\infty}^{-\infty} (-dy') \, f(x-y') \, g(y')$$

$$= \int_{-\infty}^{\infty} dy' \, g(y') \, f(x-y')$$

$$= g \otimes f$$

(b)

$$f \otimes g = \int_{-\infty}^{\infty} dx' \left\{ \delta(x' - \tfrac{d}{2}) \left\{ \Theta(x-x'+\tfrac{a}{2}) - \Theta(x-x'-\tfrac{a}{2}) \right\} \right.$$

$$\left. + \; \delta(x' + \tfrac{d}{2}) \left\{ \Theta(x-x'+\tfrac{a}{2}) - \Theta(x-x'+\tfrac{a}{2}) \right\} \right\}$$

$$= \Theta(x - \tfrac{d}{2} + \tfrac{a}{2}) - \Theta(x - \tfrac{d}{2} - \tfrac{a}{2})$$

$$+ \; \Theta(x + \tfrac{d}{2} + \tfrac{a}{2}) - \Theta(x + \tfrac{d}{2} - \tfrac{a}{2})$$

This corresponds to two slits of thickness a at $\frac{d}{2}$ & $-\frac{d}{2}$.

$\boxed{4}$

(c)

$$\mathcal{F}(f \otimes g) = \int dx \left(e^{iux} \int dx' f(x') \cdot g(x-x') dx' \right)$$

$$= \int dx' f(x') \int dx\, e^{iux}\, g(x-x')$$

$y = x - x'$

$$= \int dx' f(x') \cdot \int dy\, e^{iu(y+x')}\, g(y)$$

$$= \int dx' f(x') e^{iux'} \int dy\, g(y)\, e^{iuy}$$

$$= \mathcal{F}(f)\, \mathcal{F}(g)$$

(d) $U(u,v) \propto \mathcal{F}(g(x,y))$

$$= \mathcal{F}(g_x(x) \otimes g_y(y))$$

$$= \mathcal{F}(g(x))\, \mathcal{F}(g(y))$$

Here $\mathcal{F}(g(x))$ corresponds to N slits of length $\Delta x = a$ with separation d, and $\mathcal{F}(g(y))$ is single slit of

infinite length and thickness b. Hence

$$\mathcal{F}(g(y)) = \operatorname{sinc}^2 \frac{kb}{2} \sin \psi_y$$

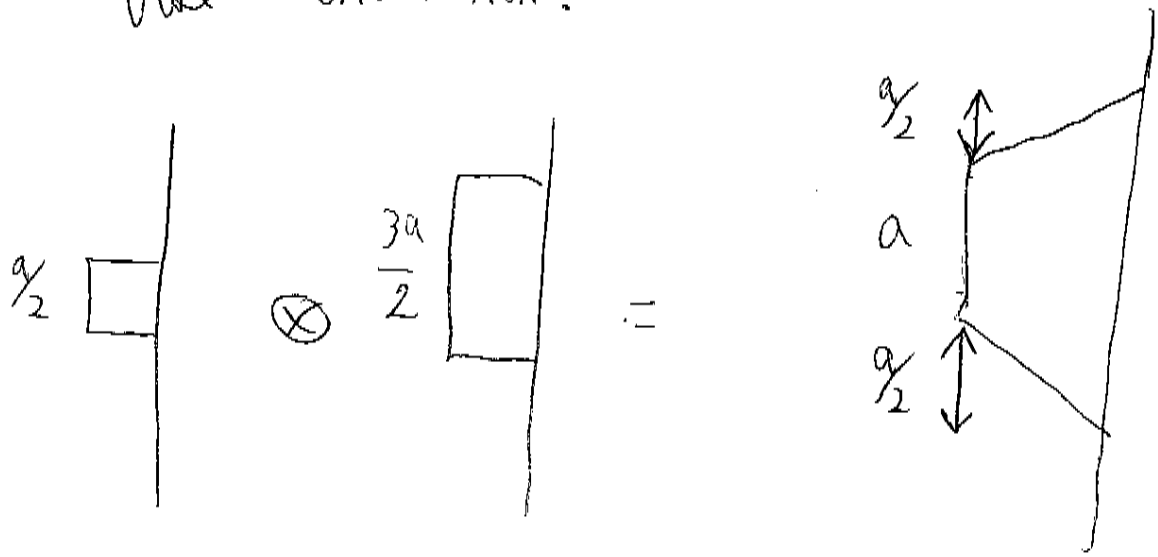
$\mathcal{F}(g(x))$ is N thick slits, so

$$N \frac{\sin^2\left(N \frac{kd}{2} \sin \psi_x\right)}{\left(N \sin \frac{kd}{2} \sin \psi_x\right)^2} \operatorname{sinc}^2 \frac{ka}{2} \sin \psi_x .$$

As a result,

$$\frac{I_N(\psi_x, \psi_y)}{I_1(0,0)} = N^2 \frac{\sin\left(N \frac{kd}{2} \sin \psi_x\right)}{\left(N \sin \frac{kd}{2} \sin \psi_x\right)^2} \operatorname{sinc}^2 \frac{ka}{2} \sin \psi_x \operatorname{sinc}^2 \frac{kb}{2} \sin \psi_y$$

(5) Use Convolution.



So via convolution,

$$\frac{I(\psi_x, \psi_y)}{I_1(0,0)} = \mathrm{sinc}^2\left(\frac{ka}{4}\sin\psi_x\right)\mathrm{sinc}^2\left(\frac{3ka}{4}\sin\psi_x\right).$$

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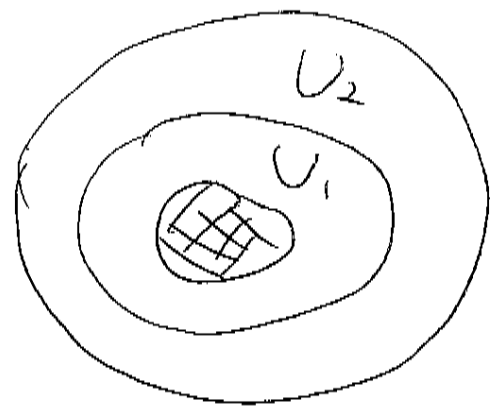
(a) For N odd zones, max occurs for

$$\frac{I}{I_0} = 4N^2 .$$ In our case, $N = 1$.

So Max is $\frac{I}{I_0} = 4$.

(b)



$U = U_1 + U_2 + \cdots$

So $U = \frac{1}{2} U_1 = U_0$; $I = I_0$.

Hence no amplification.

(3) No. $U_{ap} = 2U_{no}$ because

$U_{comp} = -U_{no}$. $U_{ap} + U_{comp} = U_{no}$ is still valid. ✓

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$$R = \sqrt{n \lambda h}.$$

Inner radius is at 1st zone, and outer radius is at 4th zone.

$$R_{inner} = \sqrt{\lambda h}$$

$$R_{out} = \sqrt{4 \lambda h}$$

So this can block zones 2 & 4 which are dominantly reducing the irradiance from inner zone. Hence max irradiance occurs with this condition and

$$|U| = |U_1| + |U_5| - |U_6| + \cdots = |U_1| + \frac{1}{2}|U_5|$$

$$= \frac{3}{2}|U_1|$$

So $$\frac{I_{max}}{I_0} = \frac{(3/2\ U_1)^2}{(1/2\ U_1)^2} = 9.$$

7

b

From Babinet's principle,

$$U_{washer} + U_{half\,1} + U_{half\,2} = U_{no}$$

$$|U_{washer}| = \frac{3}{2}|U_1| = 3|U_0|$$

$$U_{no} = U_0 \quad , \text{ and } U_{half\,1} = U_{half\,2}.$$

So $$U_{half} = \frac{1}{2}(3U_0 + U_0) = 2U_0$$

$$\rightarrow \quad I = 4I_0$$

8 P&P 22-11.

Substituting Eq (22-18), we obtain

$$\frac{\Phi(r=a)}{\Phi_{tot}} = \frac{1}{\Phi_{tot}} \iint_A \left| E_0 e^{ik(x^2+y^2)/2R(z)} e^{-(x^2+y^2)/w^2(z)} e^{i[kz+\phi(z)-\omega t]} \right| dA$$

$$= \frac{E_0^2}{\Phi_{tot}} \iint_A E_0^2(z) e^{-2(x^2+y^2)/w^2(z)} dA$$

$$= \frac{E_0^2}{\Phi_{tot}} \int_0^{2\pi} d\varphi \int_0^a r\,dr\, e^{-2r^2/w^2(z)}$$

(Following the discussion just after Eq^n (22-18), substitute $\Phi_{tot} = E_0^2(z) \frac{\pi [w(z)]^2}{2}$)

$$= E_0^2 \frac{2}{E_0^2 \pi w^2} 2\pi \frac{w^2}{4} \left(1 - e^{-2a^2/w^2}\right)$$

$$\therefore \frac{\Phi(r=a)}{\Phi_{tot}} = 1 - e^{-(2a^2)/w^2}$$