University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

PROBLEM SET 10

1.
A particle of mass m in a harmonic oscillator potential $V(x) = \frac{1}{2}m\omega_0^2 x^2$ has possible definite-energy wavefunctions $u_n(x)$ with energies $E_n = \hbar\omega_0(n + \frac{1}{2})$, where n is zero or a positive integer. The particle is in thermal equilibrium with a bath at temperature T. Its probability of having total energy E (relative to the bottom of the well) is proportional to the Boltzmann factor $e^{-E/kT}$ where k is Boltzmann's constant.

Calculate the average energy $\bar{E}$ of the particle, true for any temperature. Then take the high temperature limit $kT >> \hbar\omega_0$ and show that $\bar{E}$ reduces to kT, the classical result.

2.
Consider the potential $\frac{1}{2}m\omega^2 x^2$ of the harmonic oscillator, where ω is a constant. Define the operator $A \equiv y + iq$, where $y \equiv x\sqrt{m\omega/2}$ and $q \equiv p/\sqrt{2m\omega}$. Use the fact that $\langle x\rangle$ and $\langle p\rangle$ are physical observables so that $x^\dagger = x$ and $p^\dagger = p$, where † denotes the Hermitian conjugate. Remember that $c^\dagger = c^*$ when c is a constant. $\mathcal{H} = \mathcal{H}^\dagger$ is the Hamiltonian. For this potential, prove that

$$\begin{aligned}
A^\dagger &= y - iq \\
[A, A^\dagger] &= \hbar \\
\mathcal{H} &= \omega(q^2 + y^2) \\
\mathcal{H} &= \omega(A^\dagger A + \hbar/2) \\
\mathcal{H} &= \omega(AA^\dagger - \hbar/2) \\
[\mathcal{H}, A^\dagger] &= \hbar\omega A^\dagger \\
[\mathcal{H}, A] &= -\hbar\omega A
\end{aligned}$$

(This is the formal basis for the assertion, proved in lecture, that a harmonic oscillator has energy levels $E_n = \hbar\omega(n+\frac{1}{2})$, where n is an integer ≥ 0.)

3.
Consider the angular momentum operator $\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} = (\hbar/i)\mathbf{r} \times \nabla$. For example, $L_x \equiv (\hbar/i)\big(y(\partial/\partial z) - z(\partial/\partial y)\big)$. Define $L_\pm \equiv L_x \pm iL_y$. Prove that

$$\begin{aligned}
[L_x, L_y] &= i\hbar L_z \\
[L_y, L_z] &= i\hbar L_x \\
[L_z, L_x] &= i\hbar L_y \\
[L_+, L_-] &= 2\hbar L_z \\
[L_-, L_z] &= \hbar L_- \\
[L_+, L_z] &= -\hbar L_+ \\
[L^2, L_z] &= 0 \\
[L^2, L_\pm] &= 0 \\
L^2 &= L_-L_+ + L_z^2 + \hbar L_z \\
L^2 &= L_+L_- + L_z^2 - \hbar L_z
\end{aligned}$$

This is the formal basis for the assertion, proved in lecture, that angular momentum is quantized in integral or (for intrinsic [spin] angular momentum) *half-integral* units of $\hbar$.

4.
The spherical harmonic $Y_{lm}(\theta, \phi)$ is an eigenfunction of L^2 with eigenvalue $\hbar^2 l(l+1)$ and also of L_z with eigenvalue $\hbar m$. It is normalized so that

$$\int d\Omega\, Y_{lm}^*(\theta, \phi) Y_{lm}(\theta, \phi) = 1.$$

(a.)
Show formally that $L_+ Y_{ll} = 0$. (*Hint: evaluate* $\int d\Omega (L_+Y_{ll})^*(L_+Y_{ll})$.)
(b.)
In lecture it was proved that L_- is a lowering operator:

$$L_- Y_{lm} = C_-(l, m) Y_{l,m-1}\ ,$$

where $C_-(l, m)$ is a constant depending on l and m. Using the normality of the Y_{lm}'s, derive the value of $|C_-(l, m)|^2$.

5.
Consider the problem of an electron bound to an infinitely heavy nucleus, here using our modern understanding of orbital and spin angular momenta. Suppose that the nucleus is spinless, and

that the atom is in a state of definite *orbital* angular momentum $l = 2$. Moreover, the atom is in a state of definite projection $m_j = \frac{5}{2}$ of its *total* (spin + orbital) angular momentum on the z axis.
(a.)
What value(s) of total angular momentum quantum number j is (are) possible? Why?
(b.)
Define the expectation value of the cosine of the angle between the atom's orbital and spin angular momenta as follows:

$$\langle \cos\theta \rangle \equiv \frac{\langle \mathbf{L} \cdot \mathbf{S} \rangle}{\sqrt{\langle L^2 \rangle \langle S^2 \rangle}}.$$

Evaluate $\langle \cos\theta \rangle$ for this problem. (*Hint.* Consider $\langle J^2 \rangle$, where $\mathbf{J} = \mathbf{L} + \mathbf{S}$.)

6.
Rohlf 7.27.

7.
Rohlf 8.21.

8.
Rohlf 8.25.